2/1 B

PRO $n = 2, 3, \dots$ $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$

$$\Leftrightarrow \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right)^2 > n$$

① PRO ~~...~~ $n = 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+\sqrt{1}}{\sqrt{2}} = \sqrt{2}+1 > 1$$

②

PLATÍ

$$\left(\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} \right)^2 = \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \right)^2 = \left(\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \right) + \cancel{\dots}$$

$$+ 2 \cdot \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \cdot \frac{1}{\sqrt{n+1}} + \frac{1}{n+1} \overset{?}{>} \cancel{\frac{1}{\sqrt{n+1}}} \sqrt{n+1}$$

$$2 \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \cdot \frac{1}{\sqrt{n+1}} > 2\sqrt{n} \cdot \frac{1}{\sqrt{n+1}}$$

$\vdots$

2/1E

PRO $n \in \mathbb{N}$: $1 + \frac{n}{2} \leq \sum_{k=1}^{2^n} \frac{1}{k} \leq n + \frac{1}{2}$

DŮKAZ INDUKCÍ

① PRO $n=1$

$$1 + \frac{1}{2} \leq \frac{1}{1} + \frac{1}{2} \leq 1 + \frac{1}{2} \quad \ldots \quad ✓$$

② $n \rightarrow n+1$

A)

$$1 + \frac{n+1}{2} \leq \sum_{k=1}^{2^{n+1}} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^n+2^n} \frac{1}{k}$$

$$1 + \frac{n+1}{2} \leq 1 + \frac{n}{2} + \sum_{k=2^n+1}^{2^n+2^n} \frac{1}{k}$$

$$\frac{1}{2} < \underbrace{\sum_{k=2^n+1}^{2^n+2^n} \frac{1}{k}}_{\geq 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}}$$

2/1 F

PRO KAŽDÉ $n! = 1 \cdot 2 \cdot 3 \cdots n \leq \left(\frac{n+1}{2}\right)^n$

1. PRO $n=1$

$$1 \leq \left(\frac{1+1}{2}\right)^1$$

$$1 \leq 1$$

2. $n \Rightarrow n+1$: CHCEME $(n+1)! \leq \left(\frac{n+2}{2}\right)^{n+1}$

ZA PŘEDPOKLADU $n! \leq \left(\frac{n+1}{2}\right)^2$

$$(n+1)! = n!\,(n+1) \leq \left(\frac{n+1}{2}\right)^n \cdot (n+1) \overset{?}{\leq} \left(\frac{n+2}{2}\right)^{n+1}$$

$$\left(\frac{n+1}{2}\right)^n (n+1) \overset{?}{\leq} \left(\frac{n+2}{2}\right)^{n+1}$$

$$(n+1)^{n+1} \overset{?}{\leq} \frac{1}{2}(n+2)^{n+1} \quad / \cdot 2\left(\frac{1}{(n+1)^{(n+1)}}\right)$$

$a_n = \left(1 + \frac{1}{n}\right)^n$

$a_1 = 2 \leq a_2 \leq a_3$

$$2 \overset{?}{\leq} \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}$$

3/3 $(1+x)^n \geq (1+nx)\ ;\ x \geq -2\ ,\ n \in N$

1

$n=1\ :\ (1+x) \stackrel{?}{=} 1+x$ ✓

$n=2\ :\ (1+x)^2 \stackrel{?}{=} 1+2x$

$x^2 \geq 0$ ✓

$n \to n+2$

2 $(1+x)^n \geq (1+nx)$

$(1+x)^{n+2} \geq 1+(n+2)x$

$(1+x)^2(1+x) \geq 1+2x+nx$

$(1+2x+x)(1+x)^n => 1+2x+nx$

$(1+x)+2x(1+x)+x^2(x+1)^n \geq 1+nx+2x$

$2x(1+x)^n + x^2(1+x)^n \geq 2x$

$2(1+x)^n + x(1+x)^n \leq 2$

$(2+x)(1+x)^n \leq 2$

$\cancel{\langle -2, 0)}\ (1+x)^{2n}(2+k) \leq 2$